Endogenous Aggregate Elasticity of Substitution*

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Abstract

In the literature studying aggregate economies the aggregate elasticity of substitution (AES) between capital and labor is often treated as a constant or "deep" parameter. This view contrasts with the conjecture put forward by Arrow et al. (1961) that AES evolves over time and changes with the process of economic development. This paper evaluates this conjecture in a simple dynamic multi-sector growth model, in which AES is endogenously determined. Our findings support the conjecture, and in particular demonstrate that AES tends to be positively related to the state of economic development, a result consistent with recent empirical findings.

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"Given systematic intersectoral differences in the elasticity of substitution and income elasticity of demand, the possibility arises that the process of economic development itself might shift the over-all elasticity of substitution."

Arrow, Chenery, Minhas, and Solow (1961)

1 Introduction

More than forty years ago, Arrow, Chenery, Minhas and Solow (1961) (henceforth ACMS) put forth the idea that the aggregate elasticity of substitution (AES) between capital and labor may vary with the process of economic development. In this paper, we examine this conjecture in a multi-sector model of a growing economy, in which AES varies over time. The model demonstrates that AES tends to be positively related to the level of economic development, a result consistent with recent empirical findings.

The notion of elasticity of substitution (ES) was invented by Hicks in his seminal book, *The Theory of Wages* (1932), to analyze changes in income shares of labor and capital in a growing economy, and has since played major roles in many branches of economics. In macroeconomics and growth theory, however, researchers have frequently used the Cobb-Douglas (CD) production function to describe aggregate output behavior, thereby accepting the implication of the CD production function that ES is unitary or capital and labor income shares remain constant over time. This implication was supported by Kaldor's (1961) stylized facts and other empirical studies, convincing researchers that AES was indeed a "deep parameter" equal to unity in aggregate economies.

Nonetheless, some researchers have expressed skepticism about the assertion that AES is a "deep" parameter equal to unity. For example, Solow (1957), although perhaps the first to suggest the use of the CD function to study aggregate production, has noted that there is no evidence to support the assertion.[1] The dissatisfaction with the CD production function has led ACMS (1961) to invent a more flexible constant-elasticity-of-substitution (CES) production function. The possibility of non-unitary ES has initiated a new line of research on the role of AES in economic growth.[2]

[1]Solow (1958) pointed out that Kaldor's stylized facts had held over a short period of time for which data were available.

[2]See, e.g. Pitchford (1960), Uzawa (1962), McFadden (1963), Sato (1967), Klump and de La Grandville (2000) and Miyagiwa and Papageorgiou (2003) for theoretical investigations, and Fisher, Solow and Kearl (1977) and Yuhn (1991) for empirical works.

More recently, empirical studies have questioned the relevance of the CD function as an aggregate production function. Pereira (2002) has found that AES is non-unitary and changing over time. Masanjala and Papageorgiou (2004) have shown that Solow cross-country regressions favor the CES over the CD technology. Klump, McAdam and Willman (forthcoming), using a normalized CES function with factor-augmenting technical progress, have found that the ES is significantly below unity for the U.S. economy. Duffy and Papageorgiou (2000) have used panel data techniques to estimate an aggregate CES production function using data from 82 countries and found that the CD production function is rejected in favor of the CES specification. Furthermore, dividing the sample countries into several subsamples, Duffy and Papageorgiou (2000) have discovered that physical capital and (human capital-adjusted) labor are more substitutable in the wealthiest group of countries relative to the poorest group. This finding is of particular interest to us because it not only supports the ACMS conjecture quoted at the beginning of this paper but also establishes the direction in which the process of economic development shifts AES.

All these empirical works demonstrate that AES is not a constant but a variable. However empirical works do not explain why AES exhibits such variability across countries and time. We thus begin our analysis with an inquiry into what determines AES between capital and labor. Although there is an enormous literature dealing with ES, our knowledge of what affects the degree of substitution between capital and labor is very limited. What little we know about the behavior of AES once again traces back to Hicks (1932, 1963), who has offered the following conjectures: In a multisector economy, AES is greater, (a) the greater the intra-sectoral ES, (b) the greater the difference in factor intensity among sectors, (c) the greater the inter-commodity substitution by consumers, and (d) the greater the technological innovation that enhances intra-sectoral and inter-commodity substitution.[3] These Hicksian conjectures concern the variability of AES across economies at a given time, and are distinct from the ACMS conjecture that concerns the variability of AES across time.

In this paper we aim to construct a model that will enable us to examine the ACMS conjecture that AES changes with economic development. Such a model should be multi-sector to incorporate the Hicksian determinants of AES, and dynamic so that AES changes as the economy grows. To

[3]Only conjectures (a), (b) and (d) appear in the original 1932 edition of *The Theory of Wages*, while conjecture (c) is included in the 1963 edition.

the best of our knowledge this is the first attempt in the literature to endogenize the behavior of AES along growth paths.

Our model can now be outlined. The model economy is closed to international trade and comprises three sectors – two intermediate-good sectors and one final good sector. In each period, the existing capital and labor endowments are combined to produce the intermediate goods, which are in turn combined to produce the final good. A given fraction of the final good output then is invested to increase the next period's capital stock while the remainder is consumed. In the following period, the same process repeats itself with an increased capital stock. In a nutshell, our model is thus a static factor-endowment model grafted to the standard neoclassical growth model. The advantage of this model is that it can be solved sequentially.[4] In particular, AES in each period is determined endogenously by the existing endowments of capital and labor and their equilibrium inter-sectoral allocation.

While relatively simple, however, our model cannot in general be solved analytically. Therefore we resort to numerical analysis for the most part of the paper. Numerical results we obtain support the ACMS conjecture that AES changes over time with the process of economic development. More importantly, in all but one case we examine, AES increases as the economy grows, a result consistent with the Duffy-Papageorgiou (2000) finding.

The remainder of the paper is organized as follows. Section 2 presents the basic factor-endowment model and merges it with the Solow growth model. The general formula for AES is also derived. In Section 3 we investigate the behavior of AES along the growth path using numerical approximation techniques. Section 4 discusses our findings and concludes with suggestions for future research.

2 Model

We consider an economy in an infinite discrete-time horizon. At each period t, the economy is endowed with K_t units of capital and L_t units of labor and produces two intermediate goods (X_i,

[4] Corden (1971), Ventura (1997) and Ferreira and Trejos (2002) have also adopted similar structures in different contexts.

$i = 1, 2$) according to the constant-returns-to-scale (CRTS) production function

$$X_{it} = G_i(K_{it}, L_{it}),$$

where K_{it} and L_{it} denote, respectively, capital and labor units used in sector i. The factor markets clear when

$$\begin{aligned} L_{1t} + L_{2t} &= L_t \\ K_{1t} + K_{2t} &= K_t. \end{aligned}$$

Intermediate goods combine to produce a single final good (Y_t) according to the CRTS technology

$$Y_t = F(X_{1t}, X_{2t}). \tag{1}$$

At the aggregate level, final output must equal consumption (C_t) plus capital investment (I_t). Capital investment (I_t) adds to the next period's capital stock (K_{t+1}). The economy's feasibility constraint combined with the law of motion of capital is given by

$$I_t = K_{t+1} - (1 - \delta) K_t = Y_t - C_t,$$

where δ denotes the capital depreciation rate. In the next period, the economy solves the same production problem as above, endowed with K_{t+1} units of capital and L_{t+1} units of labor.

We next describe how to obtain AES in each period. As mentioned in the Introduction, AES depends on endowments of capital and labor and their equilibrium difference in inter-sectoral allocation. To derive the latter, we adopt the dual approach based on Jones (1965). Thus, in equilibrium unit production costs equal intermediate-good prices (p_i). (For simplicity we drop the time subscript in the following analysis.)

$$c_1(w, r) = p_1 \tag{2}$$

$$c_2(w, r) = p_2, \tag{3}$$

where w denotes the wage, r denotes the rental price and $c_i(w, r)$ is the minimum unit cost function.

We suppose that given p_i these equations uniquely determine the equilibrium factor prices (w, r). Letting $(\hat{\ })$ indicate a percentage change, i.e., $\hat{x} = dX/X$, and differentiating (2) and (3) yields

$$\theta_{1w}\hat{w} + \theta_{1r}\hat{r} = \hat{p}_1 \tag{4}$$

$$\theta_{2w}\hat{w} + \theta_{2r}\hat{r} = \hat{p}_2, \tag{5}$$

where θ_{ij} is the distributive share of factor j in sector i; for example, $\theta_{iw} = (wL_i) / (p_i X_i)$. Equations (4) and (5) imply that

$$\hat{w} - \hat{r} = (\hat{p}_1 - \hat{p}_2)\Theta^{-1}, \tag{6}$$

where $\Theta \equiv \theta_{1w}\theta_{2r} - \theta_{2w}\theta_{1r} = \theta_{1w} - \theta_{2w} = \theta_{2r} - \theta_{1r}$. Note that $\Theta > (<)\ 0$ iff good 1 is more labor (capital) intensive relative to good 2.

The factor markets clear when

$$\begin{aligned} X_1 c_{1w}(w,r) + X_2 c_{2w}(w,r) &= L \\ X_1 c_{1r}(w,r) + X_2 c_{2r}(w,r) &= K, \end{aligned}$$

where by the Shephard-Samuelson lemma

$$\begin{aligned} c_{iw}(w,r) &\equiv \partial c_i(w,r)/\partial w = L_i/X_i \\ c_{ir}(w,r) &\equiv \partial c_i(w,r)/\partial r = K_i/X_i, \end{aligned}$$

are unit factor demands. Totally differentiating the factor market-clearing conditions yields

$$\lambda_{1w}\left(\hat{X}_1 + \hat{c}_{1w}\right) + \lambda_{2w}\left(\hat{X}_2 + \hat{c}_{2w}\right) = \hat{L} \tag{7}$$

$$\lambda_{1r}\left(\hat{X}_1 + \hat{c}_{1r}\right) + \lambda_{2r}\left(\hat{X}_2 + \hat{c}_{1r}\right) = \hat{K}, \tag{8}$$

where $\lambda_{iw} = L_i/L$, and $\lambda_{ir} = K_i/K$ are sectoral factor shares.

Letting σ_i denote ES between capital and labor in sector i, we have

$$\begin{aligned} \hat{c}_{iw}(w,r) &= \left(c_{iww}dw + c_{iwr}dr\right)/c_{iw} \\ &= -\left(rc_{iwr}dw/w - c_{iwr}dr\right)/c_{iw} \\ &= -(rc_{iwr}/c_{iw})(\hat{w} - \hat{r}) \\ &= -\sigma_i\theta_{ir}(\hat{w} - \hat{r}), \end{aligned}$$

where the second equality follows because $c_{iw}(w,r)$ is homogeneous of degree zero, and the final equality follows from the definition $\sigma_i = (c_i c_{iwr})/(c_{iw}c_{ir})$, and $\theta_{\imath r} = rc_{ir}/c_i$. Similarly,

$$\hat{c}_{ir}(w,r) = \sigma_i\theta_{iw}(\hat{w} - \hat{r}).$$

Substituting theses expressions into (7) and (8), we obtain, after rearranging,

$$\lambda_{1w}\hat{X}_1 + \lambda_{2w}\hat{X}_2 = \hat{L} + b_w(\hat{w} - \hat{r}) \tag{9}$$

$$\lambda_{1r}\hat{X}_1 + \lambda_{2r}\hat{X}_2 = \hat{K} - b_r(\hat{w} - \hat{r}), \tag{10}$$

where

$$\begin{aligned} b_w &= \lambda_{1w}\theta_{1r}\sigma_1 + \lambda_{2w}\theta_{2r}\sigma_2 \\ b_r &= \lambda_{1r}\theta_{1w}\sigma_1 + \lambda_{2r}\theta_{2w}\sigma_2. \end{aligned}$$

Equations (9) and (10) yield

$$\hat{X}_1 - \hat{X}_2 = (\hat{L} - \hat{K})\Lambda^{-1} + (\hat{w} - \hat{r})(b_w + b_r)\Lambda^{-1}, \tag{11}$$

where $\Lambda = \lambda_{1w}\lambda_{2r} - \lambda_{1r}\lambda_{2w} = \lambda_{1w} - \lambda_{1r} = \lambda_{2r} - \lambda_{2w} > (<) \ 0$ iff good 1 is relatively more labor (capital) intensive.

For the final good sector, the CRTS technology implies that relative demand for the intermediate goods depends only on the relative price so

$$\hat{X}_1 - \hat{X}_2 = -\phi(\hat{p}_1 - \hat{p}_2), \tag{12}$$

where ϕ is ES between the intermediate goods.

AES, denoted by σ, is defined by

$$\sigma = -(\hat{L} - \hat{K})/(\hat{w} - \hat{r}).$$

To find σ, combine (6), (11) and (12) to obtain

$$-\phi\Theta(\hat{w} - \hat{r}) = (\hat{L} - \hat{K})\Lambda^{-1} + (\hat{w} - \hat{r})(b_w + b_r)\Lambda^{-1}.$$

Collecting terms leads to

$$\begin{aligned} \sigma &= (b_w + b_r) + \phi\Theta\Lambda \\ &= (\lambda_{1w}\theta_{1r} + \lambda_{1r}\theta_{1w})\sigma_1 + (\lambda_{2w}\theta_{2r} + \lambda_{2r}\theta_{2w})\sigma_2 + (\theta_{1w} - \theta_{2w})(\lambda_{1w} - \lambda_{1r})\phi. \end{aligned} \tag{13}$$

A number of points are worth noting here. First, σ is a function of the three primary elasticities of substitution, σ_1, σ_2, ϕ, the sectoral factor shares, $\lambda_{iw}, \lambda_{ir}$, and the sectoral factor income shares θ_{iw}, θ_{ir}, all of which are in general endogenous. Second, the coefficients of σ_1, σ, and ϕ in equation (13) sum to unity, implying that σ is the weighted average of those parameters.[5] Third, these

[5] This was first noted by Hicks (1963, p.341).

weights in general vary as factor allocations change over time, even if the sectoral elasticity terms are assumed constant. Only under the following two circumstances will σ remain constant. The first is when the sectoral and final-good elasticities of substitution are all equal ($\sigma_1 = \sigma_2 = \phi$). Ferreira and Trejos (2002) have examined a particular model of this case, where all the three goods are produced under CD technologies, so AES is unity. The second is when one intermediate good uses only capital and the other only labor. Ventura (1997) has explored such a case, where he assumed the CES technology for the final good so that AES equals the ES of the final good sector, ϕ.

In summary, the static factor-endowment model yields the equilibrium factor allocations (K_i/K, L_i/L), the equilibrium final output (Y), and AES (σ) for each period t. To recast the model in the growth context, we use the Solow (1956) model as a baseline.[6] To simplify the analysis we assume zero population growth and zero technical progress for the remainder of the paper. The discrete-time Solow model then implies the per capita growth rate given by

$$k_{t+1}/k_t - 1 = sf(k_t)/k_t - \delta, \tag{14}$$

where $k_t = K_t/L$, s is the exogenous saving rate, and $f(k_t)$ is per capita output. The main departure from the standard Solow model is that $f(k_t)$ depends on resource allocations at each period t.[7] More specifically, from the fact that all production functions (sectoral and aggregate) have CRTS, it is shown that

$$y_t = F\left[G_1(\lambda_{1r}k_t, \lambda_{1w}), G_2(\lambda_{2r}k_t, \lambda_{2w})\right].$$

where $\lambda_{ij} = \lambda_{ij}(k_t)$.

3 A Numerical Analysis

In this section we use the above model to examine the behavior of AES along the transitional path of a growing economy. Since the model is analytically intractable, we resort to a numerical analysis. To that end we suppose that both intermediate goods are produced under the CD and

[6] Though the Solow model is chosen for its simplicity, future work may consider more complicated growth models such as the optimal growth model, i.e. Cass-Koopmans (1965), and the R&D-based growth model, i.e. Romer (1990).

[7] We thank a referee for pointing out this dependence of per capita output on the sectoral distribution of inputs.

the final good under the CES technologies. This particular specification maintains the qualitative implications of the general model with the minimal computational complexity.

We now describe the specific production functions we use. The final good is produced under the following CES technology

$$Y = A(\rho)\left[\gamma(\rho) X_1^{\rho} + (1-\gamma(\rho)) X_2^{\rho}\right]^{1/\rho}, \tag{15}$$

where $\phi = 1/(1-\rho)$, $(\rho \leq 1)$, $A(\rho)$ is the "normalized" technology index, and $\gamma(\rho)$ is the "normalized" distribution parameter. The normalization of these parameters follows the procedure due to de La Grandville (1989). Without such normalization a change in ρ in the CES function not only alters the curvature of the isoquant but shifts the whole isoquant map so that comparisons of growth paths at different values of ϕ are improper.[8]

More specifically, de La Grandville (1989) suggested the following normalization procedure: Given the standard intensive-form CES production function $f(k_t) = A[\gamma k_t^q + (1-\gamma)]^{1/q}$, where the elasticity of substitution between capital and labor is $\zeta = 1/(1-q)$, $(q \leq 1)$, choose arbitrary baseline values for capital per worker $(\bar{k})$, output per worker $(\bar{y})$ and the marginal rate of substitution between capital and labor defined by $\bar{m} = [f(\bar{k}) - \bar{k} f'(\bar{k})]/f'(\bar{k})$ (primes denote derivatives). Then, use those baseline values to solve for the normalized efficiency parameter $A(q) = \bar{y}\left[(\bar{k}^{1-q} + \bar{m})/(\bar{k} + \bar{m})\right]^{1/q}$, and the normalized distribution parameter $\gamma(q) = \bar{k}^{1-q}/\left(\bar{k}^{1-q} + \bar{m}\right)$ as a function of the elasticity parameter, q.

To apply the de La Grandville (1989) normalization procedure in our model,[9] notice that (15) can be expressed as $f(x) = A(\rho)\left[\gamma(\rho)(X_1/X_2)^{\rho} + (1-\gamma(\rho))\right]^{1/\rho}$, where $y = f(x) = Y/X_2$ and $x = X_1/X_2$. Therefore, the de La Grandville (1989) normalization follows directly from choosing arbitrary baseline values for $\bar{x}$, $\bar{y}$ and the marginal rate of substitution between $X_1$and X_2, $\bar{m} = [f(\bar{x}) - \bar{x} f'(\bar{x})]/f'(\bar{x})$. The normalized efficiency parameter is then given by $A(\rho) = \bar{y}\left[(\bar{x}^{1-\rho} + \bar{m})/(\bar{x} + \bar{m})\right]^{1/\rho}$, and the normalized distribution parameter by $\gamma(\rho) = \bar{x}^{1-\rho}/\left(\bar{x}^{1-\rho} + \bar{m}\right)$.

The intermediate goods are produced under CD technologies

$$X_1 = K_1^{a_1} L_1^{1-a_1}$$

[8] For additional discussion on the normalized CES function and the "inter-family" problem associated with CES functions see Klump and de La Grandville (2000), Klump and Preissler (2000), and Saam (2005).

[9] We thank Marianne Saam for her suggestions on the issue of normalization.

$$X_2 = K_2^{a_2} L_2^{1-a_2},$$

so, $\sigma_1 = \sigma_2 = 1$, and $a_i = \theta_{ir}(0 < a_i < 1)$ is now the capital income share in sector i. Without loss of generality we assume that sector 2 is more capital-intensive than sector 1, or $a_2 > a_1$.

To evaluate σ in (13), we use the relationship that $\lambda_{1w} + \lambda_{2w} = \lambda_{1r} + \lambda_{2r} = 1$ to reduce b_w and b_r to

$$\begin{aligned} b_w &= a_2 + \lambda_{1w}(a_1 - a_2) \\ b_r &= (1 - a_2) - \lambda_{1r}(a_1 - a_2). \end{aligned}$$

Adding yields

$$b_w + b_r = 1 + (a_1 - a_2)(\lambda_{1w} - \lambda_{1r}) = 1 - \Theta\Lambda.$$

Substituting this into (13), we obtain:

$$\begin{aligned} \sigma &= (b_w + b_r) + \phi\Theta\Lambda \\ &= 1 + (a_2 - a_1)(L_1/L - K_1/K)(\phi - 1). \end{aligned} \tag{16}$$

Since $\Theta\Lambda = (a_2 - a_1)(L_1/L - K_1/K) > 0$,[10] we have that $\sigma \gtreqless 1$ iff $\phi \gtreqless 1$.

Equation (16) shows that AES depends on the factor-intensity difference, $(a_2 - a_1)$, and the final-good sector ES, ϕ, and the equilibrium inter-sectoral factor allocation, $(L_1/L - K_1/K)$. As the latter changes with capital accumulation, so does σ along the growth path. Interestingly, (16) also shows that AES is positively related to the factor-intensity difference if and only if ϕ exceeds unity.

We can now solve the model in two steps. At period t, we compute equilibrium factor allocations $(K_{1t}/K_t, L_{1t}/L)$, AES (σ_t), and final output per capita $(f(k_t))$. $f(k_t)$ is then used in equation (14) to obtain the next period's per capita and total capital endowment (k_{t+1} and K_{t+1}). At period $t+1$, given the new supply of capital (K_{t+1}) we compute $(K_{1,t+1}/K_{t+1}, L_{1,t+1}/L)$ and therefore σ_{t+1} and $f(k_{t+1})$. We repeat this process until we reach the steady state $\{k^*, \sigma^*, (K_1/K)^*, (L_1/L)^*\}$.

[Table 1 here]

Table 1 summarizes the parameter values used to carry out the numerical exercises.[11] It is

[10] Given equilibrium w and r, $a_2 - a_1 > 0$ implies $(1 - a_1)/a_1 = \theta_{1w}/\theta_{1r} > \theta_{2w}/\theta_{2r} = (1 - a_2)/a_2$. But since $\theta_{iw}/\theta_{ir} = (wL_i)/(rK_i)$, the last inequality implies $L_1/K_1 > L_2/K_2$, which implies $\lambda_{1w}\lambda_{2r} - \lambda_{1r}\lambda_{2w} \equiv \Lambda > 0$. But $\Lambda = \lambda_{1w} - \lambda_{1r} = L_1/L - K_1/K$. Thus, $a_2 - a_1 > 0$ implies $L_1/L - K_1/K > 0$.

[11] The procedure used in obtaining our numerical results is described in Appendix A.

important to state up front that those parameter values are not meant for calibration (there is lack of information for such exercise) but for numerical evaluation of the model only. Since theory gives us no guidance on how to choose values for ϕ, a_1 and a_2 we consider a wide range of values. In particular, we let ϕ range from 0.1 to 10 while letting $(a_2 - a_1)$ take either 0.5 or 0.8. In addition, we set the Solow growth model parameter values to be $s = 0.3$ and $\delta = 0.1$, and for simplicity assume no population and technology growth (i.e., $n, \mu = 0$). Our choice of the savings rate and depreciation rate is common in the growth literature.[12] Finally, we choose parameter values for $\bar{y} = \bar{x} = 2$, and $\bar{m} = 0.5$ to incorporate de La Grandville's normalization discussed above. The choice of these values, although arbitrary, helps us manage units in the quantitative exercise and provides sensible examples.

Figures 1-5 present five numerical examples in which $\phi \in \{0.1, 0.5, 1, 2, 10\}$, respectively. In each figure the left panel with charts (A, B, C) assumes the narrower factor-intensity difference i.e., $(a_1, a_2) = (0.2, 0.7)$, whereas the right panel adopts the wider difference of $(a_1, a_2) = (0.1, 0.9)$. Chart A (in Figures 1-5) illustrates per capita capital, k, along the transition path and at steady state. Regardless of parameter values the model maintains the transitional and steady-state properties of the standard Solow growth model; growth is faster at low levels of capital per worker and there is convergence to a single steady state.

[**Figures 1-5 here**]

Chart B depicts the relationship between k and sector 1's capital and labor shares (K_1/K, L_1/L). In Figure 1, as k increases towards the steady state, the difference, $(L_1/L - K_1/K)$, first widens and then narrows. In Figure 2, the difference steadily widens in the relevant range. Figure 3 depicts the benchmark case, in which ϕ is set equal to unity, and the relative factor shares remain constant throughout. Finally, in Figures 4 and 5 we see the difference in factor shares narrowing continuously.

Chart C presents our key finding; that is, AES varies with the process of economic development as conjectured by ACMS. In Figures 1-2 and 4-5, all our numerical examples except one demonstrate that σ increases as k increases along the transitional path towards steady state, corroborating the Duffy-Papageorgiou (2000) finding that AES is greater in richer countries than in poorer countries.

[12]This assumption is for computational ease and has no qualitative implications on our results.

The only exception to the positive relationship between σ and k appears in the right panel of Figure 1, where ϕ is set very high ($\phi = 10$) and the relative factor intensity is great ($a_2 - a_1 = 0.8$).

We now explain intuitively the relationship between k and σ exhibited in Chart C of Figures 1-5. Observe that σ is linearly related to $(L_1/L - K_1/K)$ by equation (16) because $(a_2 - a_1)$ and ϕ are exogenous. Therefore, to understand the relationship between σ and k we need only to explain the relationship between $(L_1/L - K_1/K)$ and k , which is depicted in Chart B of Figures 1-5.

To understand the depictions in Chart B, rewrite the difference as[13]

$$L_1/L - K_1/K = (L_2/K)(K_2/L_2 - K/L), \tag{17}$$

and equation (16) as

$$\sigma = 1 + (\phi - 1)(a_2 - a_1)(L_2/K)(K_2/L_2 - K/L).$$

Suppose now that there is a one percent increase in K due to investment. Assume for the moment that the relative intermediate goods price does not change and that intermediate-good production occurs within the diversification cone.[14] Then, by the Rybczynski theorem the second intermediate good sector expands by λ_{1w}/Λ percent and hence its labor demand rises by the same percentage under constant returns.[15] Thus, the ratio L_2/K in equation (17) increases by $\lambda_{1w}/\Lambda - 1 = \lambda_{1r}/\Lambda > 0$, raising σ.

Turning to the second term on the right-hand side of equation (17), note that K_2/L_2 is constant in the assumed absence of relative price change. Since K/L increases, the second term must decline, thereby lowering σ. It follows that, as capital accumulates at the constant relative price, the difference $(L_1/L - K_1/K)$ increases so long as the first term on the right-hand side of (17) dominates but eventually decreases when the second terms becomes dominant.

In a complete analysis, the timing of the above turnaround is modulated by the relative price changes between the intermediate goods which have so far being ignored. If ϕ is much greater than unity as in Figure 1, the two intermediate goods are strong substitutes in the final good sector, so an increase in the capital stock is absorbed mostly by changes in quantities supplied of the intermediate goods rather than by their relative price changes. With little change in the relative price, the

[13] $L_1/L - K_1/K = \lambda_{1w} - \lambda_{1r} = \lambda_{2r} - \lambda_{2w} = K_2/K - L_2/L = (L_2/K)(K_2/L_2 - K/L)$.

[14] For a discussion of the diversification cone, see, e.g. Dixit and Norman (1980).

[15] This follows from equations (9) and (10).

turnaround occurs relatively soon, as in Figure 1. Thus, σ first rises but eventually falls before reaching the steady state, as shown in the right panel of Figure 1. A similar relationship exists in the left panel but there the steady state capital per worker is lower (compare $k^* = 3.488$ to $k^* = 6.987$) so that the economy reaches the steady-state capital per worker before capital per worker reaches the turnaround level. Therefore, AES monotonically increases as the economy converges to steady state. When ϕ is smaller but still exceeds unity as in Figure 2, where $\phi = 2$, a relative price change induced by capital accumulation is more pronounced. As the price of good 1 rises relative to that of good 2, the wage rises relative to the rental by the Stolper-Samuelson theorem, inducing firms to economize on labor use. The resultant rise in the ratio K_2/L_2 mitigates the negative impact of the second term on the right of equation (17), thereby raising the turnaround level of capital per worker above the steady-state level. Thus, the difference $(L_1/L - K_1/K)$ continues to increase until k reaches the steady state value.

Finally, if ϕ is less than one, the intermediate goods are complementary in the final good production, so capital accumulation tends to expand both intermediate-good output levels. As this happens, the second intermediate good sector, which is relatively capital intensive, gets little labor from the first since the labor supply is fixed and hence the ratio L_2/K falls. With both terms in equation (17) falling, the difference $(L_1/L - K_1/K)$ narrows as shown in Chart B of Figures 4 and 5. However, with $\phi < 1$ that means σ increases as shown in Chart C of the same figures.

Our numerical analysis thus yields the following conclusion. As the economy grows from a low state of economic development, AES increases, but for ϕ exceeding unity, the rate of increase eventually turns negative when capital per worker reaches a certain critical level (for ϕ less than unity this turnaround never occurs for the reason stated above). However, in all but one case we examined this critical level of capital per worker exceeds the steady-state level so AES steadily rises as the economy converges to steady state. When is that condition violated? The answer can be inferred from the recent works of Klump and de La Granville (2000) and Klump and Preissler (2000), which have analytically demonstrated that an economy with a higher (exogenous) ES experiences a higher capital per worker in steady state. In our model equation (16) implies that AES is higher when ϕ is higher and the factor intensity difference $(a_2 - a_1)$ is greater. An economy described in the right panel of Figure 1 has the highest ϕ and takes the larger of the two factor intensity differences we consider. Therefore, in this case the steady-state capital per worker is most likely to exceed the turnaround level. It turns out that that is the only case in our examples in which AES turns around.

4 Discussion and Conclusion

The idea that the aggregate elasticity of substitution (AES) between capital and labor evolves with the process of economic development goes back to Arrow et al. (1961). To evaluate this conjecture, we have constructed a multi-sector model of economic growth, in which AES is endogenously determined and varies as the economy grows. We have then applied new modeling and numerical approximation techniques to solve this highly non-linear model. Our results support the ACMS conjecture generally. More importantly, we have shown that AES is positively related to the level of economic development.

To ensure that our results are not driven by our model specifications, parameter values and the normalization procedure, we have performed two additional experiments. Firstly, we have repeated the entire exercise using CES functions in the intermediate goods sectors, but obtained the qualitatively identical results in this CES-CES-CES setting. Secondly, we have performed another round of numerical analysis using a special case of de La Grandville normalization that was first pointed out by Kamien and Schwartz (1968) in which $\bar{x} = 1$. In this case it is easily shown that $A(\rho) = \bar{y}$, $\gamma(\rho) = \gamma$ and $\bar{m} = (1-\gamma)/\gamma$. The results of these exercises, summarized in Appendix B without charts, are again qualitatively identical to our results in Figures 1-5.[16]

Our findings have quite important implications for the empirical and theoretical literature. First, our key finding that AES varies positively with the level of development not only verifies the ACMS conjecture but also gives theoretical underpinning to the empirical findings of Duffy and Papageorgiou (2000). Our finding is also related to the notion of Variable Elasticity of Substitution (VES) pioneered by Revankar (1971) (i.e. $\sigma^{VES} = 1 + ak$). However, the two models take diametrically opposite approaches. Revankar assumes the positive relationship between capital per worker and AES at the outset, and devises an ad hoc aggregate production function that yields that relationship. We, on the other hand, obtain such a relationship endogenously from the market

[16] The detailed results and charts from these two experiments are available upon request from the authors.

equilibrium condition of a growing multi-sector economy.

Second, as mentioned earlier, Klump and de La Grandville (2000) and Klump and Preissler (2000) have recently utilized the normalized CES production function in the Solow (1956) growth model to show that a country endowed with a higher ES experiences higher capital and output per worker in transition and in steady state. Our analysis demonstrates that their result holds in a multi-sector economy. For example, recall that ϕ increases from 0.1 to 10 as we move back from Figure 5 through Figure 1. The corresponding right panels indicate that the steady-state capital stock, k^*, increases successively from 3.482 in Figure 5 to 3.958 in Figure 4, to 3.984 in Figure 3, to 4.491 in Figure 2, and finally 6.987 in Figure 1. Similar results obtain from a comparison of the left panels. In sum, our examples show that increases in σ_1, σ_2 and ϕ raise capital and output per worker in transition and in steady state.

The above theoretical works in turn give an intuition to our numerical findings. In our model AES can fall as an economy approaches its steady state, if the intermediate goods are substitutes and the steady-state level of capital per worker is less than some critical level. But the above-mentioned theoretical studies imply that the steady-state level of capital per worker is lower, the smaller the (exogenous) ES. Thus, if the initial AES is not too great, AES increases as the economy grows.

Third, we have presented a minimalist model capable of endogenizing AES. As such, it is possible to extend our model in several different directions. A good place to start is to augment the present model with substitution-enhancing technical change (possibly along the lines of Kamien and Schwartz (1969) and Acemoglu (2002)) and investigate its effects on AES and economic development. Another promising extension is to open the economy to trade and examine the relationship between the degree of openness and growth as mediated by the endogenous ES.

Fourth, important implications of AES on growth such as its effect on country convergence, its effect on per capita GDP vs. per capita capital, its contribution to the gap between poor and rich countries, are interesting directions for future research. We suggest that these issues deserve serious attention and thorough analysis perhaps in a richer growth model that can potentially be taken to the data by means of a calibration exercise.

Fifth, with the emergence of reliable empirical estimates of sectoral production parameters, we could more systematically investigate how changes in different parameters underlying AES can affect the economic development path.

Finally, we offer the following conjecture: our results are independent of the Solow growth model and robust with other growth models, in which capital accumulation is the engine of growth. The basis for our belief is that how AES changes as an economy's capital-labor ratio grows is not a property of the Solow growth model but a property of the underlying general equilibrium model. However, the steady-state level of capital per worker is a property of a growth model. Thus, use of alternative growth models may result in having the higher steady-state level of capital per worker, thereby making the monotonic rise in AES more unlikely within wider ranges of parameters. We thus expect that future work will evaluate our conjecture in richer growth models.

Appendix A

Numerical solution procedure

Numerical results are produced using MATHCAD. The programs are available upon request from the authors.

Before proceeding, note that the factor allocations of the static factor-endowment model can be replicated by the optimization problem: Maximize equation (15) subject to the technological and endowment constraints. The first-order conditions (FOCs) are:

$$\gamma(\rho)a_1 K_1^{a_1\rho-1} L_1^{(1-a_1)\rho} = (1-\gamma(\rho))\, a_2 \,(K-K_1)^{a_2\rho-1}\,(L-L_1)^{(1-a_2)\rho} \tag{A1}$$

$$\gamma(\rho)\,(1-a_1)\, K_1^{a_1\rho} L_1^{(1-a_1)\rho-1} = (1-\gamma(\rho))\,(1-a_2)\,(K-K_1)^{a_2\rho}\,(L-L_1)^{(1-a_2)\rho-1}. \tag{A2}$$

Step 1. Assign values to parameters: $A(\bar{x},\bar{y},\bar{m}), \gamma(\bar{x},\bar{y},\bar{m}), a_1, a_2, \phi, s, \delta$.

Step 2. Assign initial values to K and L. For simplicity we assume that the population is constant and equal to unity.

Step 3. Use numerical approximation methods to solve the nonlinear system of two equations (A1-A2) with two unknowns (K_1, L_1).

Step 4. Use the equilibrium allocations (K_{1t}, L_{1t}) from Step 3 in the final-good production function, equation (15), to obtain the final per capita output $(f(k_t))$, which can be expressed as

$$y_t = A(\rho)\left\{\gamma(\rho)\left[(\lambda_{1r}k_t)^{a_1}\lambda_{1w}^{1-a_1}\right]^{\rho} + (1-\gamma(\rho))\left[(\lambda_{2r}k_t)^{a_2}\lambda_{2w}^{1-a_2}\right]^{\rho}\right\}^{1/\rho}.$$

Step 5. Use the final output in the dynamic growth equation (14) to obtain the next period's stock of capital (k_{t+1}).

Step 6. The new k_{t+1} (and therefore K_{t+1}) is then used in Step 3 to close the loop. The recursive system characterized by steps 3–4–5–6–3 continues until it yields the steady-state $\{k^*, \sigma^*, (K_1/K)^*, (L_1/L)^*\}$.

Appendix B

Sensitivity analysis

In this appendix the robustness of our baseline parameter values is checked using the alternative normalization procedure proposed by Kamien and Schwartz (1968), in which we set $\bar{x} = \bar{y} = 1$ and allow $\bar{m}$ to take alternative values $\bar{m} = \{0.3, 0.5, 0.8\}$. To save space we report only results in the case where $(a_1, a_2) = (0.1, 0.9)$. The plots similar to Figures 1-5 are available from the authors.

[Table 2 here]

Panels A, B and C in Table 2 show that the results are similar to those of our model, thereby attesting the robustness of our model despite the alternative normalization procedure.

References

Acemoglu, D., 2002. Directed technical change. Review of Economic Studies 69, 781-810.

Arrow, K.J., Chenery, H.B., Minhas, B.S., Solow, R.M, 1961. Capital-labor substitution and economic efficiency. Review of Economics and Statistics 43, 225-250.

Cass, D., 1965. Optimum growth in an aggregate model of capital accumulation. Review of Economic Studies 32, 233-240.

Corden, W., 1971. The effects of trade on the rate of growth, in Bhagwati, J.N., Jones, R.W., Mundell, R., Vanek, J., eds., Trade, balance of payments and growth. North-Holland.

de La Grandville, O., 1989. In quest of the Slutsky Diamond. American Economic Review 79, 468-481.

Dixit, A.K., Norman, V., 1980. Theory of international trade. Cambridge University Press.

Duffy, J., Papageorgiou, C., 2000. A cross-country empirical investigation of the aggregate production function specification. Journal of Economic Growth 5, 87-120.

Ferreira P.C., Trejos, A., 2002. On the growth effects of barriers to trade. Working paper, Department of Economics, Northwestern University.

Fisher, M.F., Solow, R.M.,Kearl, J.M.,1977. Aggregate production functions: some CES experiments. Review of Economic Studies 44, 305-320.

Hansen, B.E., 2000. Sample splitting and threshold estimation. Econometrica 68, 575-603.

Hicks, J.R., 1932. The theory of wages, 1^{st} edition, London: MacMillan.

Hicks, J.R., 1963. The theory of wages, 2^{nd} edition, London: MacMillan.

Jones, R.W., 1965. The structure of a simple general equilibrium models. Journal of Political Economy 73, 557-572.

Kaldor, N., 1961. Capital accumulation and economic growth, in Lutz, F.A., Hague, D.C., eds., The theory of capital. New York: St. Martin's Press.

Kamien, M.I., Schwartz, N.L., 1968. Optimal 'induced' technical change. Econometrica 36, 1-17.

Kamien, M.I., Schwartz, N.L., 1969. Induced factor augmenting technical progress from a microeconomic viewpoint. Econometrica 37, 668-684.

Klump, R., McAdam, P., Willman, A., forthcoming. Factor substitution and factor augmenting technical change in the US: a normalized supply-side system approach. Review of Economics and Statistics.

Klump, R., Preissler, H., 2000. CES production functions and economic growth. Scandinavian Journal of Economics 102, 41-56.

Klump, R., de La Grandville, O., 2000. Economic growth and the elasticity of substitution: two theorems and some suggestions. American Economic Review 90, 282-291.

Koopmans, T.C., 1965. On the concept of optimal economic growth, in The econometric approach to development planning, North Holland.

Masanjala, W., Papageorgiou, C., 2004. The Solow model with CES technology: nonlinearities and parameter heterogeneity. Journal of Applied Econometrics 19, 171-201.

McFadden, D., 1963. Constant elasticity of substitution production functions. Review of Economic Studies 30, 73-83.

Miyagiwa, K., Papageorgiou C., 2003. Elasticity of substitution and growth: normalized CES in the Diamond model. Economic Theory 21, 155-165.

Pereira, C.M., 2004. Variable Elasticity of Substitution, technical change, and economic growth. Working paper, Department of Economics, North Carolina State University.

Pitchford, J.D., 1960. Growth and the elasticity of substitution. Economic Record 36, 491-504.

Revankar, N.S., 1971. A class of Variable Elasticity of Substitution production functions. Econometrica 29, 61-71.

Romer, P., 1990. Endogenous technical change. Journal of Political Economy 98, S71-S102.

Saam, M., 2005.. Essays on growth, distribution, and factor substitution. PhD Thesis, University of Frankfurt.

Sato, K., 1967. A two-level Constant-Elasticity-of-Substitution production function. Review of Economic Studies 34, 201-218.

Solow, R.M.,1956. A contribution to the theory of economic growth. Quarterly Journal of Economics 70, 65-94.

Solow, R.M., 1957. Technical change and the aggregate production function. Review of Economics and Statistics 39, 312-320.

Solow, R.M., 1958. A skeptical note on the constancy of relative shares. American Economic Review 48, 618-631.

Uzawa, H., 1962. Production functions with Constant Elasticity of Substitution. Review of Economic Studies 30, 291-299.

Ventura, J., 1997. Growth and interdependence. Quarterly Journal of Economics 112, 57-84.

Yuhn, K., 1991. Economic growth, technical change biases, and the elasticity of substitution: a test of the de La Grandville-hypothesis. Review of Economics and Statistics 73, 340-346.

Table 1: Parameter values used in the simulations for the CD-CD-CES case

$\phi \in \{0.1, 0.5, 1, 2, 10\}$	$\bar{y} = 2$	$s = 0.3$
$(a_1, a_2) = (0.2, 0.7)$	$\bar{x} = 2$	$\delta = 0.1$
$(a_1, a_2) = (0.1, 0.9)$	$\bar{m} = 0.5$	$n, \mu = 0$

Table 2: Kamien-Schwartz normalization

Panel A: $\bar{x} = \bar{y} = 1$, $\bar{m} = 0.3$

ϕ	k^*	σ^*	$(K_1/K)^*$	$(L_1/L)^*$
10	3.492	6.656	0.148	0.934
2	3.483	1.597	0.209	0.955
1	3.478	1	0.270	0.968
0.5	3.473	0.748	0.348	0.977
0.1	3.466	0.627	0.467	0.986

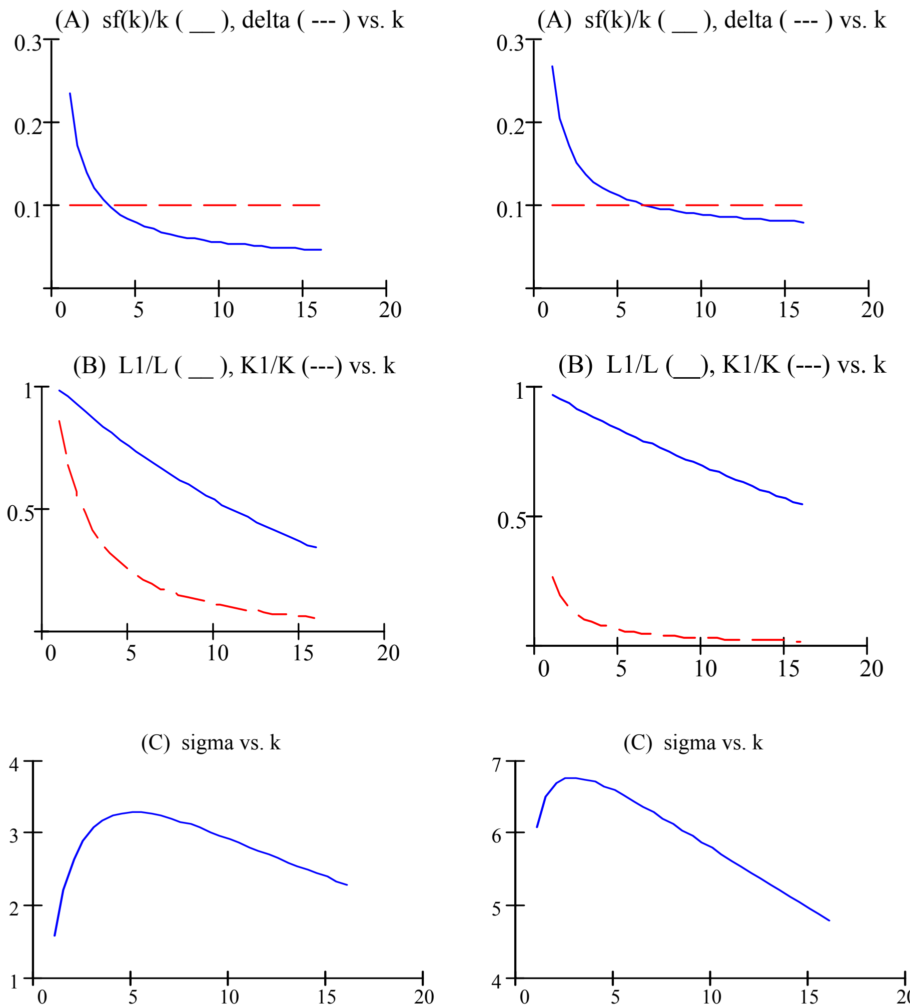
Panel B: $\bar{x} = \bar{y} = 1$, $\bar{m} = 0.5$

ϕ	k^*	σ^*	$(K_1/K)^*$	$(L_1/L)^*$
10	4.49	6.585	0.058	0.834
2	3.988	1.638	0.116	0.914
1	3.968	1	0.182	0.947
0.5	3.486	0.722	0.274	0.968
0.1	3.469	0.613	0.447	0.985

Panel C: $\bar{x} = \bar{y} = 1$, $\bar{m} = 0.8$

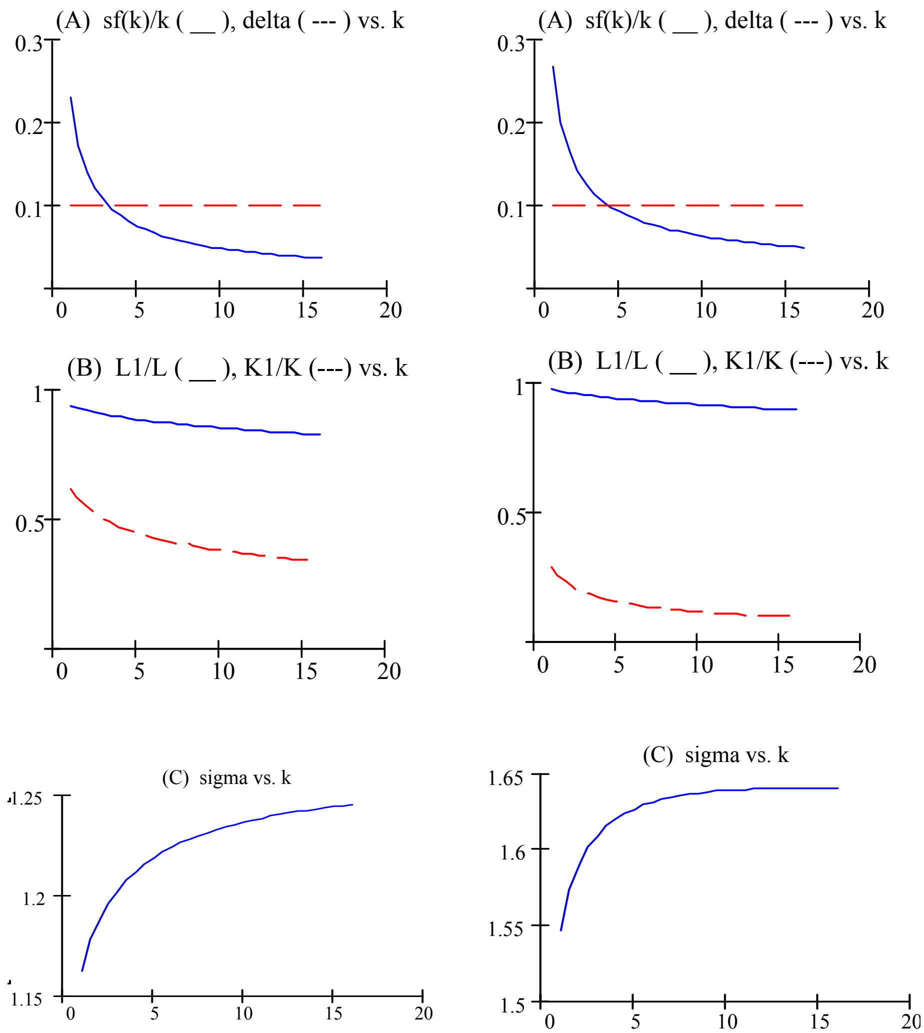
ϕ	k^*	σ^*	$(K_1/K)^*$	$(L_1/L)^*$
10	9.992	4.0175	0.009	0.428
2	5.485	1.6219	0.060	0.837
1	4.481	1	0.122	0.918
0.5	3.973	0.7067	0.226	0.959
0.1	3.474	0.5998	0.428	0.984

Figure 1: Transitional dynamics and steady state in model with CD-CD-CES technologies ($\phi = 10$)
Left panel: $(a_1, a_2) = (0.2, 0.7)$ Right panel: $(a_1, a_2) = (0.1, 0.9)$



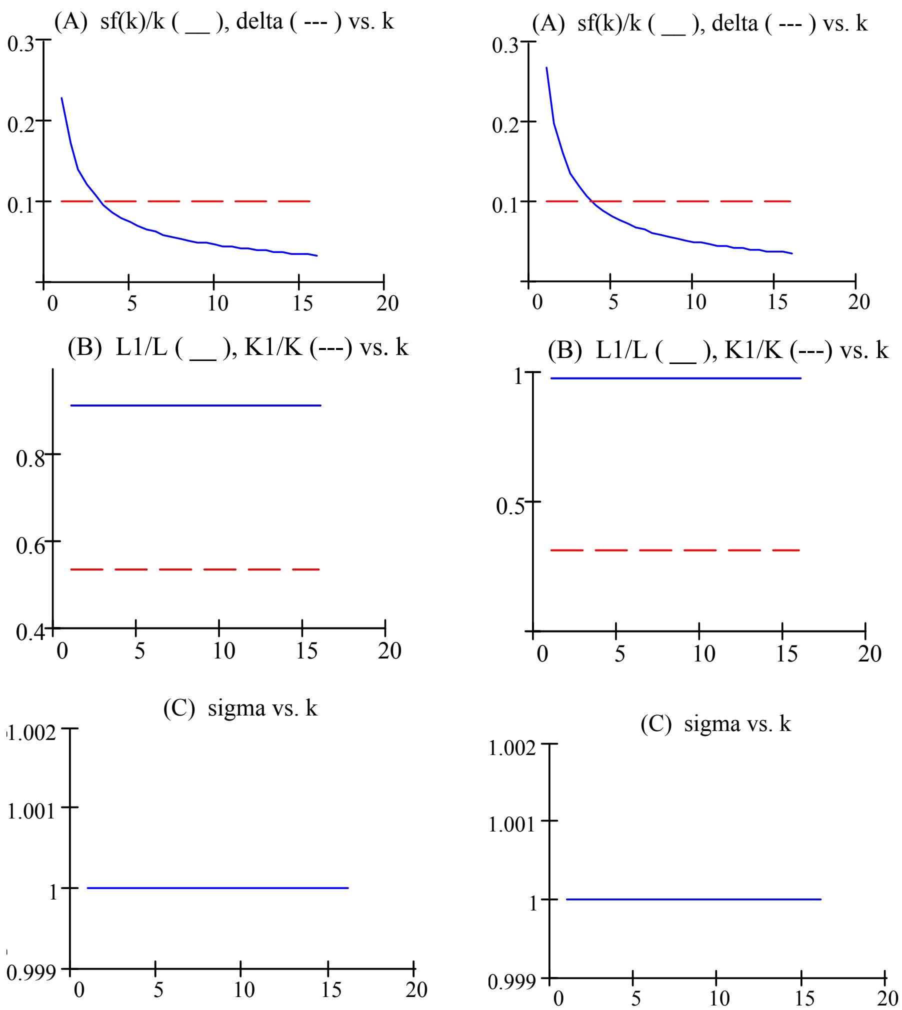
Notes: The illustrations above are constructed using a MATHCAD numerical solver. We assume the following parameter values: $s = 0.3, \delta = 0.1, \bar{m} = 0.5, \bar{k} = 2, \bar{y} = 2$. The following steady-state values are obtained. Left panel: $k^* = 3.488, \sigma^* = 3.176, (K_1/K)^* = 0.351, (L_1/L)^* = 0.835$. Right panel: $k^* = 6.987, \sigma^* = 6.279, (K_1/K)^* = 0.041, (L_1/L)^* = 0.774$.

Figure 2: Transitional dynamics and steady state in model with CD-CD-CES technologies ($\phi = 2$)
Left panel: $(a_1, a_2) = (0.2, 0.7)$ Right panel: $(a_1, a_2) = (0.1, 0.9)$



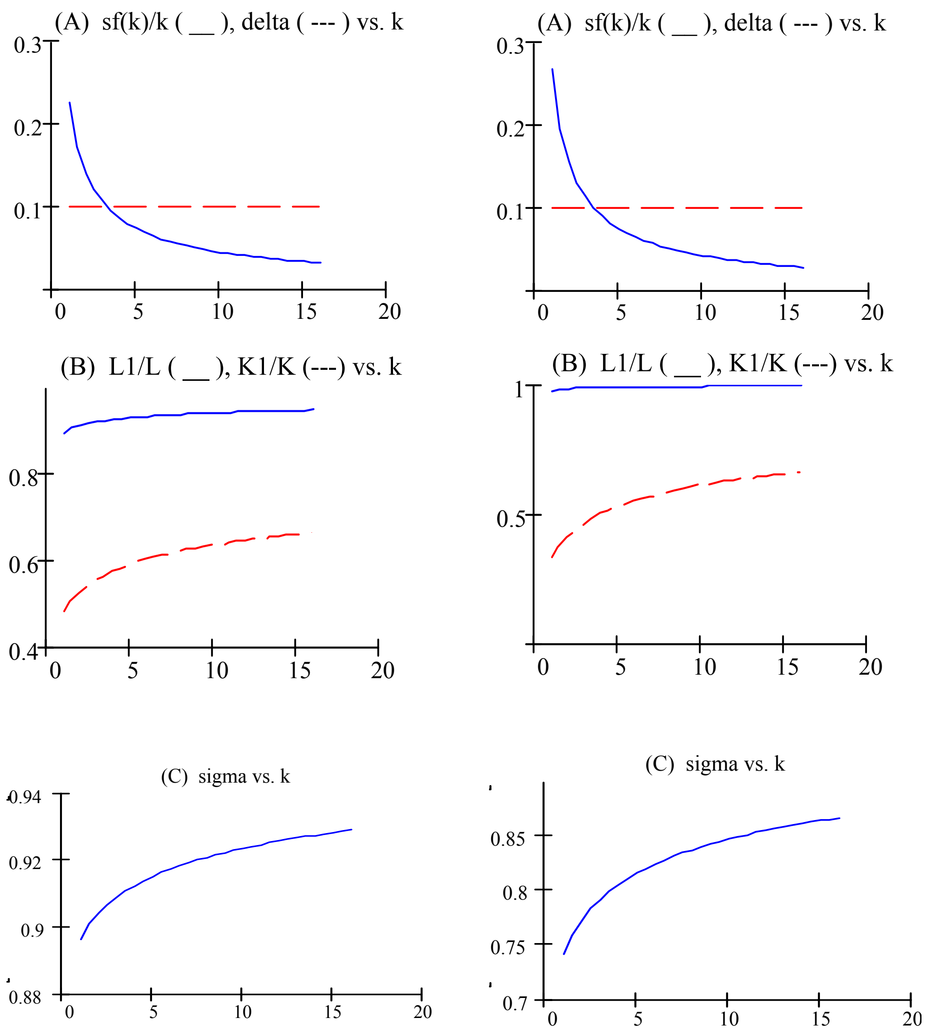
Notes: The illustrations above are constructed using a MATHCAD numerical solver. We assume the following parameter values: $s = 0.3, \delta = 0.1, \bar{m} = 0.5, \bar{k} = 2, \bar{y} = 2$. The following steady-state values are obtained. Left panel: $k^* = 3.484, \sigma^* = 1.207, (K_1/K)^* = 0.482, (L_1/L)^* = 0.897$. Right panel: $k^* = 4.491, \sigma^* = 1.624, (K_1/K)^* = 0.160, (L_1/L)^* = 0.939$.

Figure 3: Transitional dynamics and steady state in model with CD-CD-CES technologiesl ($\phi = 1$)
Left panel: $(a_1, a_2) = (0.2, 0.7)$ Right panel: $(a_1, a_2) = (0.1, 0.9)$



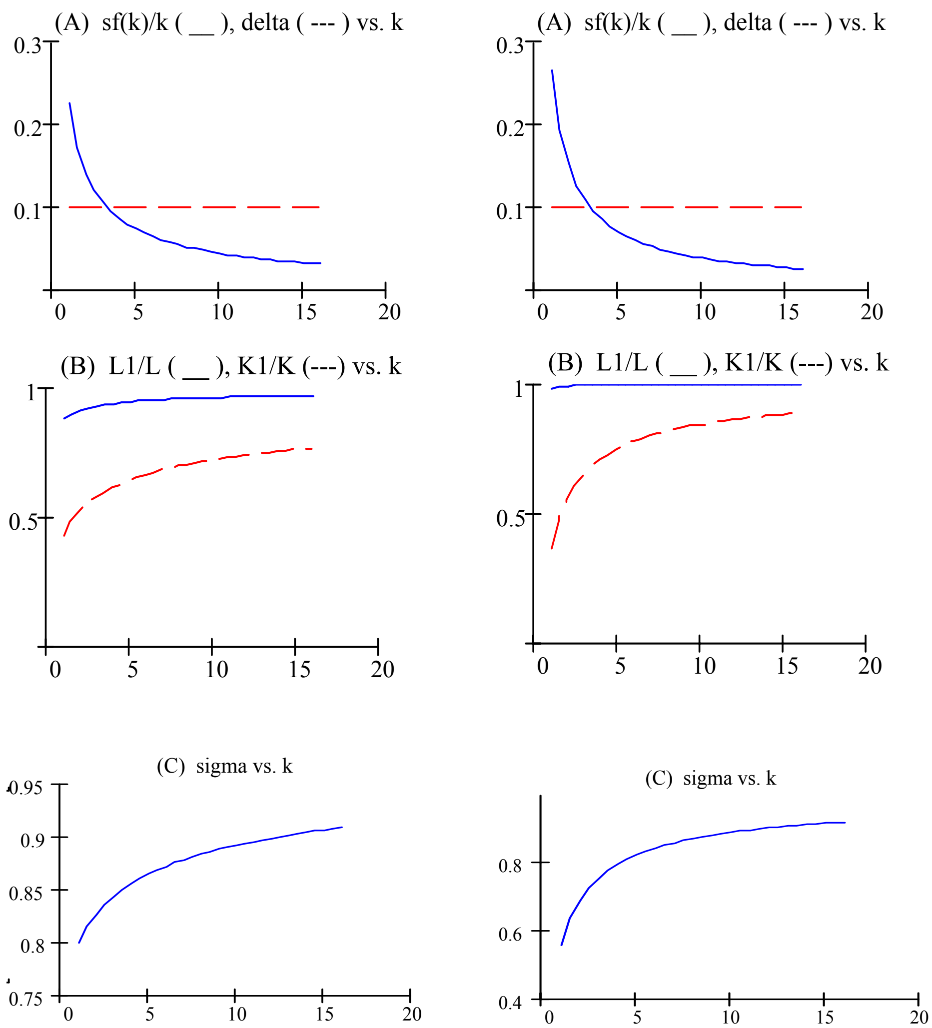
Notes: The illustrations above are constructed using a MATHCAD numerical solver. We assume the following parameter values: $s = 0.3, \delta = 0.1, \bar{m} = 0.5, \bar{k} = 2, \bar{y} = 2$. The following steady-state values are obtained. Left panel: $k^* = 3.482, \sigma^* = 1, (K_1/K)^* = 0.533, (L_1/L)^* = 0.914$. Right panel: $k^* = 3.984, \sigma^* = 1, (K_1/K)^* = 0.308, (L_1/L)^* = 0.973$.

Figure 4: Transitional dynamics and steady state in model with CD-CD-CES technologiesl ($\phi = 0.5$)
Left panel: $(a_1, a_2) = (0.2, 0.7)$ Right panel: $(a_1, a_2) = (0.1, 0.9)$



Notes: The illustrations above are constructed using a MATHCAD numerical solver. We assume the following parameter values: $s = 0.3, \delta = 0.1, \bar{m} = 0.5, \bar{k} = 2, \bar{y} = 2$. The following steady-state values are obtained. Left panel: $k^* = 3.482, \sigma^* = 0.911, (K_1/K)^* = 0.567, (L_1/L)^* = 0.924$. Right panel: $k^* = 3.958, \sigma^* = 0.805, (K_1/K)^* = 0.501, (L_1/L)^* = 0.988$.

Figure 5: Transitional dynamics and steady state in model with CD-CD-CES technologies ($\phi = 0.1$)
Left panel: $(a_1, a_2) = (0.2, 0.7)$ Right panel: $(a_1, a_2) = (0.1, 0.9)$



Notes: The illustrations above are constructed using a MATHCAD numerical solver. We assume the following parameter values: $s = 0.3, \delta = 0.1, \bar{m} = 0.5, \bar{k} = 2, \bar{y} = 2$. The following steady-state values are obtained. Left panel: $k^* = 3.480, \sigma^* = 0.849, (K_1/K)^* = 0.598, (L_1/L)^* = 0.933$. Right panel: $k^* = 3.482, \sigma^* = 0.778, (K_1/K)^* = 0.686, (L_1/L)^* = 0.994$.